Darboux integrability and invariant algebraic curves for planar polynomial systems

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Abstract
In this paper we study the normal forms of polynomial systems having a set of given generic invariant algebraic curves.

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1. Introduction and statement of the main results

Nonlinear ordinary differential equations appear in many branches of applied mathematics and physics. In this paper we only consider autonomous differential systems. For a two-dimensional system the existence of a first integral completely determines its phase portrait. Of course, the easiest planar integrable systems are the Hamiltonian ones. The planar integrable systems which are not Hamiltonian can be in general very difficult to detect. Many different methods have been used for studying the existence of first integrals for non-Hamiltonian systems based on: Noether symmetries [5], the Darboux theory of integrability [14], the Lie symmetries [7, 23], the Painlevé analysis [2], the use of Lax pairs [19], the direct method [16, 17], the linear compatibility analysis method [27], the Carlemann embedding procedure [1, 6], the quasimonomial formalism [3], etc.

The algebraic theory of integrability is a classical one, which is related to the first part of the Hilbert's 16th problem. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial systems and the number of invariant algebraic curves they have (see Darboux [14] and Poincaré [24]).

Jouanolou [18] extended the planar Darboux theory of integrability to polynomial systems in $\mathbb{R}^n$ or $\mathbb{C}^n$, for extension to other fields see [28]. In [4, 9, 10, 20], the authors developed the Darboux theory of integrability essentially in $\mathbb{R}^2$ or $\mathbb{C}^2$ considering not only the invariant

algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves.

Prelle and Singer [25], using methods of differential algebra, showed that if a polynomial vector field has an elementary first integral, then it can be computed using the Darboux theory of integrability. Singer [26] proved that if a polynomial vector field has Liouvillian first integrals, then it has integrating factors given by Darbouxian functions. Some related results can be found in [8].

In this paper we are mainly interested in the polynomial differential systems which have a given set of invariant algebraic curves, independent of whether they are integrable or not. Thus, first we study the normal forms of planar polynomial vector fields having a given set of generic invariant algebraic curves. That is, in some sense we are interested in a kind of inverse theory of the Darboux theory of integrability.

In this work we deal with the following planar (differentiable) polynomial system of degree m:

$$\dot{x} = P(x, y) \qquad \dot{y} = Q(x, y) \tag{1}$$

where $P, Q \in \mathbb{C}_m[x, y]$, the set of complex polynomials of degree at most m in the variables x and y, and $\max\{\deg P, \deg Q\} = m$.

Let $C(x, y) \in \mathbb{C}[x, y]$, the ring of complex polynomials in x and y. The algebraic curve $C(x, y) = 0$ of $\mathbb{C}^2$ is called an *invariant algebraic curve* of system (1) if

$$PC_x + QC_y = KC \tag{2}$$

for some complex polynomial $K(x, y)$, which is called the *cofactor* of $C = 0$. Here and after, we denote by C_x and C_y the derivatives of C with respect to x and y, respectively. For simplicity, in what follows we will talk about the curve $C = 0$, only saying the curve C.

We mention that the cofactor of the invariant algebraic curve for system (1) has degree at most $m - 1$. Let $C = \prod_{i=1}^{l} C_i^{n_i}$ be the irreducible decomposition of C. Then C is an invariant algebraic curve with a cofactor K of system (1) if and only if C_i is an invariant algebraic curve with a cofactor K_i of system (1). Moreover, we have $K = \sum_{i=1}^{l} n_i K_i$. For a proof see [13].

Let $F(x, y) = \exp(G(x, y)/H(x, y))$ with $G, H \in \mathbb{C}[x, y]$ coprime (or equivalently, $(G, H) = 1$). We say that F is an *exponential factor* if

$$PF_x + QF_y = LF \tag{3}$$

for some polynomial $L \in \mathbb{C}_{m-1}[x, y]$, which is called a *cofactor* of F.

Let U be an open subset of $\mathbb{C}^2$. A complex function $\mathcal{H} : U \to \mathbb{C}$ is a *first integral* of system (1), if it is constant on all solution curves $(x(t), y(t))$ of system (1), i.e., $\mathcal{H}(x(t), y(t)) \equiv$ constant for all values of t for which the solution $(x(t), y(t))$ is defined on U. If the first integral $\mathcal{H}$ is differentiable, then $P\mathcal{H}_x + Q\mathcal{H}_y = 0$ in U.

If there exists a smooth function $\mathcal{R}(x, y)$ such that $(P\mathcal{R})_x + (Q\mathcal{R})_y = 0$, then $\mathcal{R}$ is called an *integrating factor* of system (1).

If system (1) has a first integral or an integrating factor of the form

$$C_1^{\lambda_1} \cdots C_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \tag{4}$$

where C_i and F_j are the invariant algebraic curve and exponential factor of system (1) respectively and $\lambda_i, \mu_j \in \mathbb{C}$, then system (1) is called *Darboux integrable*. This kind of function is called a *Darbouxian function*.

Our first result is the following theorem.

Theorem 1. *Let $C_i = 0$ for $i = 1, \ldots, p$, be irreducible invariant algebraic curves in $\mathbb{C}^2$, and set $r = \sum_{i=1}^{p} \deg C_i$. We assume that all C_i satisfy the following generic conditions:*

(i) *There are no points at which C_i and its first derivatives all vanish.*

(ii) *The highest order terms of C_i have no repeated factors.*

(iii) *If two curves intersect at a point in the finite plane, they are transversal at this point.*

(iv) *There are no more than two curves $C_i = 0$ meeting at any point in the finite plane.*

(v) *There are no two curves having a common factor in the highest order terms.*

Then any polynomial vector field X of degree m tangent to all $C_i = 0$ satisfies one of the following statements.

(a) *If $r < m + 1$ then*

$$X = \left(\prod_{i=1}^{p} C_i \right) Y + \sum_{i=1}^{p} h_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} C_j \right) X_{C_i} \tag{5}$$

where $X_{C_i} = (-C_{iy}, C_{ix})$ is a Hamiltonian vector field, the h_i are polynomials of degree no more than $m - r + 1$, and the Y is a polynomial vector field of degree no more than $m - r$.

(b) *If $r = m + 1$ then*

$$X = \sum_{i=1}^{p} \alpha_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} C_j \right) X_{C_i} \tag{6}$$

with $\alpha_i \in \mathbb{C}$.

(c) *If $r > m + 1$ then $X = 0$.*

This theorem, attributed to Christopher [10], was stated in several papers without proof such as [10, 12], and used in other papers [4, 21]. The proof that we present here of theorem 1 circulated as the preprint [11] but was never published. Zholadek in [29] (see also theorem 3 of [30]) stated a similar result to our theorem 1, but as far as we know the paper [29] has not been published. In any case Zholadek's approach to theorem 1 is analytical, while our approach is completely algebraic.

Statement (b) of this theorem has a corollary proposed by Christopher and Kooij [12] showing that system (6) has the integrating factor $\mathcal{R} = \left(\prod_{i=1}^{p} C_i \right)^{-1}$, and consequently the system is Darboux integrable.

The following result shows that the generic conditions of theorem 1 are necessary.

Theorem 2. *If one of the conditions (i)–(v) of theorem 1 is not satisfied, then the statements of theorem 1 do not hold.*

We have mentioned that system (1) satisfying the five assumptions of theorem 1 with $r = m + 1$ are Darboux integrable. Now we provide two examples of polynomial systems satisfying all assumptions of theorem 1 with $r = m + 1$ except either (ii) or (iii) and which are not Darboux integrable. Until now there are very few proofs of polynomial systems which are not Darboux integrable, see for instance Jouanolou [18] and Maciejewski *et al* [22].

Consider the following quadratic systems:

$$\begin{aligned} \dot{x} &= y(ax - by + b) + x^2 + y^2 - 1 \\ \dot{y} &= bx(y - 1) + a(y^2 - 1) \end{aligned} \tag{7}$$

which have the invariant circle $C_1 = x^2 + y^2 - 1 = 0$ with cofactor $K_1 = 2(x + ay)$ and the invariant straight line $C_2 = y - 1 = 0$ with cofactor $K_2 = bx + ay + a$. We note that C_1 and C_2 are tangent at the point $(0, 1)$.

Theorem 3. *There are values of the parameters a and b for which system (7) is not Darboux integrable.*

As a corollary the following result shows that there are polynomial systems with an invariant algebraic curve whose highest order term has repeated factors such that they are not Darboux integrable. Consider the following quadratic system:

$$\dot{x} = (1 - b)(x^2 + 2y - 1) - (ax - b)(y - 1) = P(x, y)$$
$$\dot{y} = -(bx + 2ay - a)(y - 1) = Q(x, y) \tag{8}$$

which has the invariant algebraic curves $C_1 = x^2 + 2y - 1 = 0$ with cofactor $K_1 = 2[(1 - b)x - ay + a]$ and $C_2 = y - 1 = 0$ with cofactor $K_2 = -(bx + 2ay - a)$. We note that the highest order term of C_1 has a repeated factor x.

Corollary 4. *There exist values of the parameters a and b for which system (8) is not Darboux integrable.*

The paper is organized as follows. In sections 2 and 3 we prove theorems 1 and 2, respectively. The proofs of theorem 3 and corollary 4 are given in sections 4 and 5, respectively.

2. Proof of theorem 1.

In the proof of this theorem we will use intensively Hilbert's nullstellensatz (see, for instance, [15]):

Set $A, B_i \in \mathbb{C}[x, y]$ for $i = 1, \ldots, r$. If A vanishes in $\mathbb{C}^2$ whenever the polynomials B_i vanish simultaneously, then there exist polynomials $M_i \in \mathbb{C}[x, y]$ and a non-negative integer n such that $A^n = \sum_{i=1}^{r} M_i B_i$. In particular, if all B_i have no common zero, then there exist polynomials M_i such that $\sum_{i=1}^{r} M_i B_i = 1$.

In what follows if we have a polynomial A we will denotes its degree by a. If we do not say anything we denote by C^c the homogeneous part of degree c for the polynomial C. We shall need the following result.

Lemma 5. *If C^c has no repeated factors, then $(C_x, C_y) = 1$.*

Proof. Suppose that $(C_x, C_y) \neq 1$. Then there exists a polynomial A nonconstant such that $A|C_x$ and $A|C_y$. Here $A|C_x$ means that the polynomial A divides the polynomial C_x. Therefore, $A^a|(C^c)_x$ and $A^a|(C^c)_y$. By the Euler theorem for homogeneous polynomials we have that $x(C^c)_x + y(C^c)_y = cC^c$. So $A^a|C^c$. Since $A^a, (C^c)_x, (C^c)_y$ and C^c are homogeneous polynomials of $\mathbb{C}[x, y]$ and A^a divides $(C^c)_x, (C^c)_y$ and C^c, the linear factors of A^a having multiplicity m, must be linear factors of C^c having multiplicity $m + 1$. This last statement follows easily identifying the linear factors of the homogeneous polynomial $C^c(x, y)$ in two variables with the roots of the polynomial $C^c(1, z)$ in the variable z. Hence, A^a is a repeated factor of C^c. It is in contradiction with the assumption. □

We first consider the case that system (1) has a given invariant algebraic curve.

Lemma 6. *Assume that polynomial system (1) of degree m has an invariant algebraic curve $C = 0$ of degree c, and that C satisfies condition (i) of theorem 1.*

(a) If $(C_x, C_y) = 1$, then system (1) has the following normal form:

$$\dot{x} = AC - DC_y \qquad \dot{y} = BC + DC_x \qquad (9)$$

where A, B and D are suitable polynomials.

(b) If C satisfies condition (ii) of theorem 1, then system (1) has the normal form (9) with $a, b \leqslant m - c$ and $d \leqslant m - c + 1$. Moreover, if the highest order term C^c of C does not have the factors x and y, then $a \leqslant p - c$, $b \leqslant q - c$ and $d \leqslant \min\{p, q\} - c + 1$.

Proof. (a) Since there are no points at which C, C_x and C_y vanish simultaneously, from Hilbert's nullstellensatz we obtain that there exist polynomials E, F and G such that

$$EC_x + FC_y + GC = 1. \qquad (10)$$

As C satisfies equation (2), we get from (2) and (10) that

$$K = (KE + GP)C_x + (KF + GQ)C_y.$$

Substituting K into (2), we get

$$[P - (KE + GP)C]C_x = -[Q - (KF + GQ)C]C_y.$$

Since $(C_x, C_y) = 1$, there exists a polynomial D such that

$$P - (KE + GP)C = -DC_y \qquad Q - (KF + GQ)C = DC_x.$$

This proves that system (1) has the form (9) with $A = KE + GP$ and $Q = KF + GQ$.

(b) From (a) and lemma 5 we get that system (1) has the normal form (9). Without loss of generality we can assume that $p \leqslant q$.

We first consider the case that C^c has neither factor x nor y. So we have $(C^c, (C^c)_x) = 1$ and $(C^c, (C^c)_y) = 1$, where $(C^c)_x$ denotes the derivative of C^c with respect to x. In (9) we assume that $a > p - c$, otherwise the statement follows. Then $d = a + 1$. Moreover, from the highest order terms of (9) we get

$$A^a C^c = D^{a+1} C_y^{c-1}$$

where C_y^{c-1} denotes the homogeneous part with degree $c - 1$ of C_y. Since $(C^c, C_y^{c-1}) = 1$, there exists a polynomial F such that

$$A^a = FC_y^{c-1} \qquad D^{a+1} = FC^c.$$

In (9) we replace A by $A - FC_y$ and D by $D - FC$, so the degrees of polynomials under consideration reduce by one. We continue this process and do the same for $\dot{y}$ until we reach a system of the form

$$\dot{x} = AC - DC_y \qquad \dot{y} = BC + EC_x \qquad (11)$$

with $a \leqslant p - c$, $d \leqslant p - c + 1$, $b \leqslant q - c$ and $e \leqslant q - c + 1$. Since $C = 0$ is an invariant algebraic curve of (11), from (2) we get

$$C(AC_x + BC_y) + C_x C_y(E - D) = KC.$$

This implies that there exists a polynomial R such that $E - D = RC$, because C with C_x and C_y are coprime.

If $e \geqslant d$, then $r = e - c$. We write $BC + EC_x = (B + RC_x)C + DC_x$ and denote $B + RC_x$ again by B, then system (11) has the form (9) where A, B and D have the required degrees.

If $e < d$, then $r = d - c$. We write $AC - DC_y = (A + RC_y)C - EC_y$ and denote $A + RC_y$ again by A, then system (11) has the form (9) where A, B and E instead of D have the required degrees. This proves the second part of (b).

Now we prove the first part of (*b*). We note that even though C^c has no repeated factor, C^c with C_x^{c-1} or C_y^{c-1} may have a common factor in x or y (for example, $C^3 = x(x^2 + y^2)$, $C^3 = y(x^2 + y^2)$ or $C^4 = xy(x^2 + y^2)$). In order to avoid this difficulty we rotate system (1) slightly such that C^c has no factors in x and y. Then, applying the above method to the new system we get that the new system has a normal form (9) with the degrees of A, B and D as those of the second part of (*b*).

We claim that under affine changes system (9) preserves its form and the upper bound of the polynomials, i.e. $a, b \leqslant m - c$ and $d \leqslant m - c + 1$. Indeed, using the affine change of variables $u = a_1 x + b_1 y + c_1$ and $v = a_2 x + b_2 y + c_2$ with $a_1 b_2 - a_2 b_1 \neq 0$, system (9) becomes

$$\dot{u} = (a_1 A + b_1 B)C - (a_1 b_2 - a_2 b_1)DC_v \qquad \dot{v} = (a_2 A + b_2 B)C + (a_1 b_2 - a_2 b_1)DC_u.$$

Hence, the claim follows. This completes the proof of (*b*), and consequently we have the proof of the lemma. $\qquad\square$

Lemma 7. *Assume that $C = 0$ and $D = 0$ are different irreducible invariant algebraic curves of system (1) of degree m, and that they satisfy conditions (i) and (iii) of theorem 1.*

(a) If $(C_x, C_y) = 1$ and $(D_x, D_y) = 1$, then system (1) has the normal form

$$\dot{x} = ACD - EC_y D - FCD_y \qquad \dot{y} = BCD + EC_x D + FCD_x \qquad (12)$$

(b) If C and D satisfy conditions (ii) and (v), then system (1) has the normal form (12) with $a, b \leqslant m - c - d$ and $e, f \leqslant m - c - d + 1$.

Proof. Since $(C, D) = 1$, the curves C and D have finitely many intersection points. By assumption (i) at each of such points there is at least one nonzero first derivative of both C and D. In a similar way to the proof of the claim inside the proof of lemma 6, we can prove that under an affine change of the variables, system (12) preserves its form and the bound for the degrees of A, B, E and F. So, we rotate system (1) slightly such that all first derivatives of C and D are not equal to zero at the intersection points.

From Hilbert's nullstellensatz, there exist polynomials M_i, N_i and R_i, $i = 1, 2$ such that

$$M_1 C + N_1 D + R_1 D_y = 1 \qquad M_2 C + N_2 D + R_2 C_y = 1. \qquad (13)$$

By lemma 6 we get that

$$P = A_1 C - E_1 C_y = G_1 D - F_1 D_y \qquad (14)$$

for some polynomials A_1, E_1, G_1 and F_1. Moreover, using the first equation of (13) we have $F_1 = SC + TD + UC_y$ for some polynomials S, T and U. Substituting F_1 into (14) we obtain that

$$(A_1 + SD_y)C + (-G_1 + TD_y)D + (-E_1 + UD_y)C_y = 0. \qquad (15)$$

Using the second equation of (13) and (15) to eliminate C_y we get

$$-E_1 + UD_y = VC + WD \qquad (16)$$

for some polynomials V and W. Substituting (16) into (15), we have

$$(A_1 + SD_y + VC_y)C = (G_1 - TD_y - WC_y)D.$$

Since $(C, D) = 1$, there exists a polynomial K such that

$$A_1 + SD_y + VC_y = KD \qquad G_1 - TD_y - WC_y = KC. \qquad (17)$$

Substituting E_1 of (16) and A_1 of (17) into (14), then we have

$$P = KCD - SCD_y + WC_y D - UC_y D_y. \qquad (18)$$

Similarly, we can prove that there exist some polynomials K', S', W' and U' such that

$$Q = K'CD + S'CD_x - W'C_xD + U'C_xD_x. \tag{19}$$

Since C is an invariant algebraic curve of (1), we have $PC_x + QC_y = K_CC$ for some polynomial K_C. Using (18) and (19) we get

$$K_CC = C[D(KC_x + K'C_y) - SC_xD_y + S'C_yD_x] + C_xC_y[D(W - W') - UD_y + U'D_x].$$

As C, C_x and C_y are coprime, there exists a polynomial Z such that

$$D(W - W') - UD_y + U'D_x = ZC. \tag{20}$$

Substituting the expression $DW - UD_y$ into (18), we get

$$P = KCD - SCD_y + W'C_yD - U'C_yD_x + ZCC_y. \tag{21}$$

Since $D = 0$ is an invariant algebraic curve of system (1), we have $PD_x + QD_y = K_DD$ for some polynomial K_D. Using (19) and (21) we get

$$K_DD = D[C(KD_x + K'D_y) + W'(C_yD_x - C_xD_y)] + D_x[CD_y(-S + S') + U'(C_xD_y - C_yD_x) + ZCC_y].$$

As D and D_x are coprime, there exists a polynomial M such that

$$CD_y(-S + S') + U'(C_xD_y - C_yD_x) + ZCC_y = MD. \tag{22}$$

The curves C and D being transversal implies that C, D and $C_xD_y - C_yD_x$ have no common zeros. From Hilbert's nullstellensatz, there exist some polynomials M_3, N_3 and R_3 such that

$$M_3C + N_3D + R_3(C_xD_y - C_yD_x) = 1. \tag{23}$$

Eliminating the term $C_xD_y - C_yD_x$ from (22) and (23), we obtain that $U' = IC + JD$ for some polynomials I and J. Hence, equation (22) becomes

$$C[I(C_xD_y - C_yD_x) + D_y(-S + S') + ZC_y] + D[J(C_xD_y - C_yD_x) - M] = 0.$$

Since $(C, D) = 1$, there exists a polynomial G such that

$$M = J(C_xD_y - C_yD_x) + GC$$
$$I(C_xD_y - C_yD_x) + D_y(-S + S') + ZC_y = GD.$$

Substituting $ZC_y - SD_y$ and U' into (21) we obtain that

$$P = (K + G)CD - (IC_x + S')CD_y + (W' - JD_x)DC_y.$$

This means that P can be expressed in the form (18) with $U = 0$.

Working in a similar way, we can express Q in the form (19) with $U' = 0$. Thus, (20) is reduced to $D(W - W') = ZC$. Hence, we have $W = W' + HC$ for some polynomial H. Consequently, $Z = HD$. Therefore, from (22) we obtain that $CD_y(-S + S') = D(M - HCC_y)$. Since $(C, D) = 1$ and $(D, D_y) = 1$, we have $S = S' + LD$ for some polynomial L. Substituting W and S into (18) we obtain that P and Q have the form (12). This proves statement (a).

As in the proof of lemma 6 we can prove that under suitable affine change of variables the form of system (12) and the bound of the degrees of the polynomials A, B, E and F are invariant. So, without loss of generality we can assume that the highest order terms of C and D are neither divisible by x nor y.

By the assumptions, the conditions of statement (a) hold, so we get that system (1) has the form (12). If the bounds of the degrees of A, B, E and F are not satisfied, we have by (12) that

$$A^aC^cD^d - E^eC_y^{c-1}D^d - F^fC^cD_y^{d-1} = 0$$
$$B^bC^cD^d + E^eC_x^{c-1}D^d + F^fC^cD_x^{d-1} = 0. \tag{24}$$

We remark that if one of the numbers $a + c + d$, $e + c - 1 + d$ and $f + c + d - 1$ is less than the other two, then its corresponding term in the first equation of (24) is equal to zero. The same remark is applied to the second equation of (24). From the hypotheses it follows that C^c and C_y^{c-1} are coprime, and also D^d and D_y^{d-1}, and C^c and D^d, respectively. Hence, from these last two equations we obtain that there exist polynomials K and L such that $E^e = KC^c$, $F^f = LD^d$ and

$$A^a = KC_y^{c-1} + LD_y^{d-1} \qquad B^b = -KC_x^{c-1} - LD_x^{d-1}.$$

We rewrite equation (12) as

$$\dot{x} = (A - KC_y - LD_y)CD - (E - KC)C_y D - (F - LD)CD_y$$
$$\dot{y} = (B + KC_x + LD_x)CD + (E - KC)C_x D + (F - LD)CD_x.$$

Thus, we reduce the degrees of A, B, E and F in (12) by one. We can continue this process until the bounds are reached. This completes the proof of statement (*b*). □

Lemma 8. *Let $C_i = 0$ for $i = 1, \ldots, p$ be different irreducible invariant algebraic curves of system (1) with $\deg C_i = c_i$. Assume that C_i satisfy conditions (i), (iii) and (iv) of theorem 1. Then*

(a) If $(C_{ix}, C_{iy}) = 1$ for $i = 1, \ldots, p$, then system (1) has the normal form

$$\dot{x} = \left(B - \sum_{i=1}^{p} \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{p} C_i \qquad \dot{y} = \left(D + \sum_{i=1}^{p} \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{p} C_i \quad (25)$$

where B, D and A_i are suitable polynomials.

(b) If C_i satisfy conditions (ii) and (v) of theorem 1, then system (1) has the normal form (25) with $b, d \leqslant m - \sum_{i=1}^{p} c_i$ and $a_i \leqslant m - \sum_{i=1}^{p} c_i + 1$.

Proof. We use induction to prove this lemma. By lemmas 6 and 7 we assume that for any l with $2 \leqslant l < p$ we have

$$P = \sum_{i=1}^{l} \left(B_i - \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{l} C_i \qquad Q = \sum_{i=1}^{l} \left(D_i + \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{l} C_i$$

where $\sum_{i=1}^{l} B_i = B$ and $\sum_{i=1}^{l} D_i = D$. Since $C_{l+1} = 0$ is an invariant algebraic curve, from lemma 6 we get that there exist some polynomials E, G and H such that

$$P = \sum_{i=1}^{l} \left(B_i - \frac{A_i C_{iy}}{C_i} \right) \prod_{i=1}^{l} C_i = EC_{l+1} - GC_{l+1,y}$$
$$Q = \sum_{i=1}^{l} \left(D_i + \frac{A_i C_{ix}}{C_i} \right) \prod_{i=1}^{l} C_i = HC_{l+1} + GC_{l+1,x}. \qquad (26)$$

Now we consider the curves

$$K_j = \prod_{\substack{i=1 \\ i \neq j}}^{l} C_i = 0 \qquad j = 1, \ldots, l.$$

From the assumptions we obtain that there are no points at which all the curves $K_i = 0$ and $C_{l+1} = 0$ intersect. Otherwise, at least three of the curves $C_i = 0$ for $i = 1, \ldots, l + 1$ intersect at some point. Hence, there exist polynomials U and V_i for $i = 1, \ldots, l$ such that

$$UC_{l+1} + \sum_{i=1}^{l} V_i K_i = 1. \qquad (27)$$

Using this equality, we can rearrange (26) as

$$(E - GUC_{l+1,y})C_{l+1} = \sum_{i=1}^{l}(B_i C_i - A_i C_{iy} + GV_i C_{l+1,y})K_i$$

$$(H + GUC_{l+1,x})C_{l+1} = \sum_{i=1}^{l}(D_i C_i + A_i C_{ix} - GV_i C_{l+1,x})K_i. \tag{28}$$

Using (27) and (28) to eliminate C_{l+1} we obtain that

$$E - GUC_{l+1,y} = \sum_{i=1}^{l} I_i K_i \qquad H + GUC_{l+1,x} = \sum_{i=1}^{l} J_i K_i$$

for some polynomials I_i and J_i. Substituting these last equalities into (28), we have

$$\sum_{i=1}^{l}(B_i C_i - A_i C_{iy} + GV_i C_{l+1,y} - I_i C_{l+1})K_i = 0$$

$$\sum_{i=1}^{l}(D_i C_i + A_i C_{ix} - GV_i C_{l+1,x} - J_i C_{l+1})K_i = 0. \tag{29}$$

It is easy to check that the expressions multiplying K_i in the two summations of (29) are divisible by C_i. Hence, there exist polynomials L_i and F_i for $i = 1, \ldots, l$ such that

$$B_i C_i - A_i C_{iy} + GV_i C_{l+1,y} - I_i C_{l+1} = L_i C_i$$

$$D_i C_i + A_i C_{ix} - GV_i C_{l+1,x} - J_i C_{l+1} = F_i C_i. \tag{30}$$

So, from (29) we get that $\sum_{i=1}^{l} L_i = 0$ and $\sum_{i=1}^{l} F_i = 0$. This implies that (26) can be rewritten as

$$P = \sum_{i=1}^{l}((B_i - L_i)C_i - A_i C_{iy})K_i \qquad Q = \sum_{i=1}^{l}((C_i - F_i)C_i + A_i C_{ix})K_i. \tag{31}$$

Moreover, we write (30) in the form

$$(B_i - L_i)C_i - A_i C_{iy} = I_i C_{l+1} - GV_i C_{l+1,y} = P_i$$

$$(D_i - F_i)C_i + A_i C_{ix} = J_i C_{l+1} + GV_i C_{l+1,x} = Q_i. \tag{32}$$

It is easy to see that C_i and C_{l+1} are invariant algebraic curves of the system $\dot{x} = P_i$, $\dot{y} = Q_i$. So, from statement (a) of lemma 7 we can obtain that

$$P_i = (B_i - L_i)C_i - A_i C_{iy} = X_i C_i C_{l+1} - Y_i C_{iy} C_{l+1} - N_i C_i C_{l+1,y}$$

$$Q_i = (D_i - F_i)C_i + A_i C_{ix} = Z_i C_i C_{l+1} + Y_i C_{ix} C_{l+1} + N_i C_i C_{l+1,x}.$$

Substituting these last two equations into (31), we obtain that system (1) has the form (25) with the $l + 1$ invariant algebraic curves $C_1, \ldots, C_{l+1}$. From induction we have finished the proof of statement (a).

The proof of statement (b) is almost identical with those of lemma 7(b), so we shall omit it here. Hence, this ends the proof of the lemma. $\square$

Proof of theorem. From lemma 8 it follows statement (a) of theorem 1.

By checking the degrees of polynomials A_i, B and D in statement (b) of lemma 8 we obtain statement (b) of theorem 1.

From statement (a) of lemma 8, we can rearrange system (1) such that it has the form (25). But from statement (b) of lemma 8 we must have $B = 0$, $D = 0$ and $A_i = 0$. This proves statement (c) of theorem 1. $\square$

3. Proof of theorem 2.

The proof is formed by the following examples. First, we consider the case $r < m + 1$. That is, the sum of degrees of the given invariant algebraic curves is less than the degree of the system plus one.

Example 1. The algebraic curve $C = y^3 + x^3 - x^2 = 0$ satisfies all conditions of theorem 1 excepting (i). The cubic system

$$\dot{x} = 2x - 2x^3 - 3xy^2 + y^3 \qquad \dot{y} = \tfrac{4}{3}y + x^2 - 3x^2y - 3y^3 \qquad (33)$$

has C as an invariant algebraic curve. We claim that system (33) does not have the form (5). Otherwise, it can be written in the form

$$\dot{x} = A(y^3 + x^3 - x^2) + D(-3y^2)$$

$$\dot{y} = B(y^3 + x^3 - x^2) + D(3x^2 - 2x)$$

where A, B and D are polynomials. It is in contradiction with (33), because in the first equation of (33) there is a linear term.

Example 2. The algebraic curve $C = y - x^2 = 0$ satisfies all conditions of theorem 1 excepting (ii). The polynomial system of degree m with $m \geqslant 2$:

$$\dot{x} = D(y) + xE(y) + A(x, y)(y - x^2) \qquad \dot{y} = 2xD(y) + 2yE(y) + B(x, y)(y - x^2) \quad (34)$$

has C as an invariant algebraic curve, where deg D, deg $E = m-1$, and deg A, deg $B \leqslant m-2$. We can write system (34) in the form (5), i.e.

$$\dot{x} = A(x, y)(y - x^2) + D(y) + xE(y)$$

$$\dot{y} = (B(x, y) + 2E(y))(y - x^2) + 2x(D(y) + xE(y)).$$

But then $\deg(B(x, y) + 2E(y)) = m - 1 > m - \deg C$.

Example 3. The algebraic curves $C_1 = x^2 + y^2 - 1 = 0$ and $C_2 = y - 1 = 0$ satisfy all conditions of theorem 1 excepting (iii). The cubic system

$$\dot{x} = -1 - y + x^2 + xy + y^2 + x^2y + y^3 = P \qquad \dot{y} = (y + x^2 + y^2)(y - 1) = Q \qquad (35)$$

has C_1 and C_2 as invariant algebraic curves. We claim that system (35) cannot be written in the form (5). Otherwise, Q can be written as

$$Q = B(x^2 + y^2 - 1)(y - 1) + D2x(y - 1)$$

where B and D are polynomials. However, there do not exist polynomials B and D such that

$$B(x^2 + y^2 - 1) + 2xD = y + x^2 + y^2. \qquad (36)$$

Because if the equality holds, then B must contain the monomial $-y$. Let ay^t be the monomial of B with the highest degree $t \geqslant 1$ and without the variable x. Then the left-hand side of (36) contains the monomial ay^{t+2}. It is in contradiction with the right-hand side of (36).

Example 4. The algebraic curves $C_1 = x = 0$, $C_2 = y = 0$ and $C_3 = x + y = 0$ satisfy all conditions of theorem 1 excepting (iv). The cubic system

$$\dot{x} = (1 + x + y + x^2 + xy)x = P \qquad \dot{y} = (1 + x^2 + 2xy + y^2)y = Q \qquad (37)$$

has these three curves as invariant algebraic curves. We claim that system (37) cannot be written in the form (5). Otherwise, the polynomial Q can be written as

$$Q = Bxy(x + y) + Dy(x + y) + Exy$$

where B, D and E are polynomials. But it is in contradiction with (37).

Example 5. The algebraic curves $C_1 = xy + 1 = 0$ and $C_2 = y = 0$ satisfy all conditions of theorem 1 excepting (v). The cubic system

$$\dot{x} = 1 + x + y - x^2 + x^3 + 2xy^2 = P \qquad \dot{y} = (x + y - x^2 + xy - y^2)y = Q \qquad (38)$$

has C_1 and C_2 as invariant algebraic curves. If we write this system in the form (5), then we have

$$Q = B(xy + 1)y + Dy^2$$

where B and C are polynomials. Comparing it with (38), we get that B cannot be a constant. So, $\deg B > 0 = m - r$, which is in contradiction with statement (a) of theorem 1.

Next, we consider the case $r = m + 1$. That is, the sum of the degrees of the given invariant algebraic curves is equal to the degree of the system plus one.

Example 6. The curve $C = x^2 + x^3 + y^3 = 0$ satisfies all conditions of theorem 1 excepting (i). The quadratic systems with C as an invariant algebraic curve can be written as

$$\dot{x} = \tfrac{3}{2}ax + \tfrac{3}{2}ax^2 - by^2 \qquad \dot{y} = \tfrac{2}{3}bx + ay + bx^2 + \tfrac{3}{2}axy$$

where a and b are arbitrary complex numbers. Obviously, if $a \neq 0$ this system cannot have the form (6).

Example 7. The curve $C = y - x^3 = 0$ satisfies all conditions of theorem 1 excepting (ii). It is an invariant algebraic curve of the system

$$\dot{x} = 1 + x - x^2 + xy \qquad \dot{y} = 3y + 3x^2 - 3xy + 3y^2.$$

This system cannot be written in the form (6).

Example 8. The curves $C_1 = x^2 + y^2 - 1 = 0$ and $C_2 = y - 1 = 0$ satisfy all conditions of theorem 1 excepting (iii). Moreover, C_1 and C_2 are invariant algebraic curves of system (7). However, system (7) does not have the form (6) if $a \neq 0$.

Example 9. The curves $C_1 = x + iy = 0$, $C_2 = x - iy = 0$ and $C_3 = x = 0$ satisfy all conditions of theorem 1 excepting (iv). The quadratic system

$$\dot{x} = -b(x^2 + y^2) + x + y(ax + by) \qquad \dot{y} = k(x^2 + y^2) + y - x(ax + by)$$

has C_1, C_2 and C_3 as invariant algebraic curves, but this system cannot take the form (6).

Example 10. The curves $C_1 = xy - 1 = 0$ and $C_2 = x = 0$ satisfy all conditions of theorem 1 excepting (v). They are invariant algebraic curves of the system

$$\dot{x} = (1 - 2x + y)x \qquad \dot{y} = 1 - y + xy - y^2.$$

Obviously, this system does not have the form (6).

Last we give the counterexamples for the case $r > m + 1$. That is, the sum of the degrees of the invariant algebraic curves is larger than the degree of the system plus one.

Example 11. The algebraic curve $C = x^4 + x^3 + y^4 = 0$ satisfies all conditions of theorem 1 excepting (i). The quadratic systems having C as an invariant algebraic curve are

$$\dot{x} = ax + ax^2 \qquad \dot{y} = \tfrac{3}{4}ay + axy.$$

So, statement (c) of theorem 1 is not satisfied.

Example 12. The algebraic curve $C = y - x^4 = 0$ satisfies all conditions of theorem 1 excepting (ii). The quadratic systems having C as an invariant algebraic curve are

$$\dot{x} = ax + bx^2 + cxy \qquad \dot{y} = 4ay + 4bxy + 4cy^2.$$

They are not zero unless $a = b = c = 0$.

Example 13. The algebraic curves $C_1 = x^2 + y^2 - 1 = 0$, $C_2 = y - 1 = 0$ and $C_3 = 4x + 3y + 5 = 0$ satisfy all conditions of theorem 1 excepting (iii). However, the quadratic system $\dot{x} = y(2x - y + 1) + x^2 + y^2 - 1$, $\dot{y} = x(y - 1) + 2y^2 - 2$ has these three curves as invariant algebraic curves.

Example 14. The algebraic curves $C_1 = x = 0$, $C_2 = y = 0$ and $C_3 = x + y = 0$ satisfy all conditions of theorem 1 excepting (iv). The linear systems having these three curves as invariant algebraic curves are $\dot{x} = ax$, $\dot{y} = ay$. They are not zero unless $a = 0$.

Example 15. The algebraic curves $C_1 = xy - 1$, $C_2 = y$ and $C_3 = y + 1$ satisfy all conditions of theorem 1 excepting (v). The quadratic system with C_1, C_2 and C_3 as invariant algebraic curves are

$$\dot{x} = a - bx - (a + b)xy \qquad \dot{y} = by(y + 1).$$

They are not zero unless $a = b = 0$.

From these 15 examples follows the proof of theorem 2. □

4. Proof of theorem 3.

The proof is separated into three parts. The first part shows that there exists a set Ω_1 of values of the parameters a and b such that system (7) has only the given two invariant algebraic curves. The second part give a proof that there exists a set Ω_2 of values of a and b such that systems (7) have no exponential factors. Moreover, $\Omega_1 \cap \Omega_2 \neq \emptyset$. The last step contributes to prove that system (7) is not Darboux integrable for $a, b \in \Omega_1 \cap \Omega_2$.

We need the following result (for a proof, see [10]).

Lemma 9. *Assume that system (1) with degree m has an invariant algebraic curve C of degree n. Let C_n, P_m and Q_m be the homogeneous parts of C with degree n, P and Q with degree m. Then the irreducible factor of C_n divides $yP_m - xQ_m$.*

The first part is formed by the following proposition, which is related to the existence of invariant algebraic curves of system (7).

Proposition 10. *For each $b \neq 1 \pm \frac{1}{p}$ with $p \in \mathbb{N}$ there exists a numerable set Υ such that if $a \in \mathbb{R} \backslash (\Upsilon \cup \{0\})$, then system (7) has no irreducible invariant algebraic curves different from $f_1 = 0$ and $f_2 = 0$.*

Proof. Assume that $C = \sum_{i=0}^{n} C_i(x, y) = 0$ is an invariant algebraic curve of system (7) with cofactor $K = K_1 + K_0$, where C_i and K_i are homogeneous polynomials of degree i. From the definition of the invariant algebraic curve, i.e. (2), we have

$$[x^2 + axy + (1 - b)y^2 + by - 1]\sum_{i=1}^{n} C_{ix} + [bxy + ay^2 - bx - a]\sum_{i=1}^{n} C_{iy} = (K_1 + K_0)\sum_{i=0}^{n} C_i.$$

Equating the terms with the same degree we obtain

$$L[C_{n-i}] = K_1 C_{n-i} + K_0 C_{n-i+1} - byC_{n-i+1,x} + bxC_{n-i+1,y} + C_{n-i+2,x} + aC_{n-i+2,y}$$
$$i = 0, 1, \ldots, n+2 \tag{39}$$

where $C_i = 0$ for $i < 0$ and $i > n$, and L is the partial differential operator

$$L = [x^2 + axy + (1-b)y^2]\frac{\partial}{\partial x} + [bxy + ay^2]\frac{\partial}{\partial y}.$$

For system (7) we have $yP_2 - xQ_2 = (1-b)y(x^2 + y^2)$. So, from lemma 9 we can assume that

$$C_n = (x^2 + y^2)^l y^m \qquad n = 2l + m.$$

Substituting C_n into (39) with $i = 0$ and doing some computations we get

$$K_1 = (2l + mb)x + a(2l + m)y.$$

Set $C_{n-1} = \sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^i$. Substituting C_{n-1}, C_n and K_1 into (39) with $i = 1$ and doing some calculations, we obtain

$$\sum_{i=0}^{n-1}(m - 1 - i + ib - mb)c_{n-1-i}x^{2l+m-i}y^i - \sum_{i=0}^{n-1} ac_{n-1-i}x^{2l+m-1-i}y^{i+1}$$

$$+ \sum_{i=0}^{n-1}(2l + m - 1 - i)(1-b)c_{n-1-i}x^{2l+m-2-i}y^{i+2}$$

$$= K_0(x^2 + y^2)^l y^m + mbx(x^2 + y^2)^l y^{m-1}$$

$$= \sum_{i=0}^{l} K_0 \binom{l}{i} x^{2l-2i} y^{m+2i} + \sum_{i=0}^{l} mb \binom{l}{i} x^{2l+1-2i} y^{m+2i-1}.$$

This equation can be written as

$$\sum_{i=0}^{n}[(m - 1 - i + ib - mb)c_{n-1-i} - ac_{n-i} + (2l + m + 1 - i)(1-b)c_{n+1-i}]x^{2l+m-i}y^i$$

$$= \sum_{i=0}^{l} K_0 \binom{l}{i} x^{2l-2i} y^{m+2i} + \sum_{i=0}^{l} mb \binom{l}{i} x^{2l+1-2i} y^{m+2i-1}$$

where $c_i = 0$ for $i < 0$ and $i > n - 1$. Equating the coefficients of $x^i y^j$ in the above equation, we get

$$[m - i - 1 + (i - m)b]c_{2l+m-1-i} - ac_{2l+m-i} + (2l + m + 1 - i)(1-b)c_{2l+m+1-i} = 0$$
$$i = 0, 1, \ldots, m - 2 \tag{40}$$

$$[(2i - 1)b - 2i]c_{2l-2i} - ac_{2l+1-2i} - (2l + 2 - 2i)(b - 1)c_{2l+2-2i} = mb \binom{l}{i}$$
$$i = 0, 1, \ldots, l \tag{41}$$

$$[2i(b - 1) - 1]c_{2l-2i-1} - ac_{2l-2i} - (2l + 1 - 2i)(b - 1)c_{2l+1-2i} = K_0 \binom{l}{i}$$
$$i = 0, 1, \ldots, l. \tag{42}$$

From the assumptions and (40) we can prove easily that $c_{2l+j} = 0$ for $j = 1, \ldots, m-1$. Equations (41) and (42) can be written as

$$c_{2l-2i} = \frac{1}{2i(b-1)-b}\left[ac_{2l+1-2i} + (2l+2-2i)(b-1)c_{2l+2-2i} + mb\binom{l}{i}\right]$$

$$c_{2l-2i-1} = \frac{1}{2i(b-1)-1}\left[ac_{2l-2i} + (2l+1-2i)(b-1)c_{2l+1-2i} + K_0\binom{l}{i}\right]$$

(43)

with $i = 0, 1, \ldots, l$. It is easy to check that

$$c_{2l} = -m \qquad c_{2l-1} = am - K_0.$$

From (43) with $i = 1$ we get that

$$c_{2l-2} = \frac{a}{b-2}(am - K_0) - ml = \mathcal{B}_1(a, b, l)(am - K_0) - m\binom{l}{1}$$

$$c_{2l-3} = \left[\frac{a^2}{(2b-3)(b-2)} + l - \frac{b-1}{2b-3}\right](am - K_0) = \mathcal{B}_2(a, b, l)(am - K_0).$$

In what follows we use induction to find the coefficients c_{2l-i} for $i = 4, \ldots, 2l$. Assume that for $i = h$ we have

$$c_{2l-2h} = \mathcal{B}_{2h-1}(a, b, l)(am - K_0) - m\binom{l}{h} \qquad c_{2l-1-2h} = \mathcal{B}_{2h}(a, b, l)(am - K_0)$$

where $\mathcal{B}_{j-1}(a, b, l)$ for $j = 2h, 2h+1$, are polynomials in a where coefficients are functions of b and l and the highest order terms of the form

$$a^{j-1}\left/\prod_{i=2}^{j}[(i-1)b - i]\right..$$

(44)

Then from (43) with $i = h + 1$ we get

$$c_{2l-2h-2} = \frac{1}{2(h+1)(b-1)-b}\left\{a\mathcal{B}_{2h}(a, b, l)(am - K_0) + (2l - 2h)(b-1)\right.$$

$$\times\left[\mathcal{B}_{2h-1}(a, b, l)(am - K_0) - m\binom{l}{h}\right] + mb\binom{l}{h+1}\right\}$$

$$= \mathcal{B}_{2h+1}(a, b, l)(am - K_0) - m\binom{l}{h+1}$$

$$c_{2l-2h-3} = \frac{1}{2(h+1)(b-1)-1}\left\{a\left[\mathcal{B}_{2h+1}(a, b, l)(am - K_0) - m\binom{l}{h+1}\right]\right.$$

$$\left. + (2l - 2h - 1)(b-1)\mathcal{B}_{2h}(a, b, l)(am - K_0) + K_0\binom{l}{h+1}\right\}$$

$$= \mathcal{B}_{2h+2}(a, b, l)(am - K_0).$$

where

$$\mathcal{B}_{2h+1} = \frac{1}{2(h+1)(b-1)-b}[a\mathcal{B}_{2h} + (2l - 2h)(b-1)\mathcal{B}_{2h-1}]$$

$$\mathcal{B}_{2h+2} = \frac{1}{2(h+1)(b-1)-1}\left[a\mathcal{B}_{2h+1} - \binom{l}{h+1} + (2l - 2h - 1)(b-1)\mathcal{B}_{2h}\right]$$

are polynomials in a of degree $2h+1$ and $2h+2$ respectively, in which the highest order terms are the form (44) for $j = 2h+2$ and $j = 2h+3$, respectively. Hence, from (43) and using induction we obtain that for $h = 0, 1, \ldots, 2l$

$$c_{2l-h} = \mathcal{B}_{h-1}(a, b, l)(am - K_0) + \frac{(-1)^{h+1} - 1}{2}m\binom{l}{h/2}.$$

Moreover, from the first equation of (43) with $i = l$, i.e. $ac_0 + (b - 1)c_1 + K_0 = 0$ we get

$$a[\mathcal{B}_{2l-1}(am - K_0) - m] + (b - 1)\mathcal{B}_{2l-2}(am - K_0) = K_0.$$

This means that

$$[a\mathcal{B}_{2l-1} + (b - 1)\mathcal{B}_{2l-2} - 1](am - K_0) = 0.$$

Since $a\mathcal{B}_{2l-1} + (b - 1)\mathcal{B}_{2l-2} - 1$ is a polynomial of degree $2l$ in the variable a, it has at most $2l$ real roots, denoted by S_l the set of the roots. Then, for $a \in \mathbb{R} \backslash S_l$ we must have $K_0 = am$.

Obviously, $\Upsilon = \cup_{l=1}^{\infty} S_l$ is a numerable set. Moreover, for each $a \in \mathbb{R} \backslash \Upsilon$ and $l \in \mathbb{N}$ we have $K_0 = am$. So, if C is an invariant algebraic curve of the above form, it has the cofactor $K = K_1 + K_0 = (2l + mb)x + a(2l + m)y + am = 2(x + ay)l + (bx + ay + a)m$.

Moreover, we can check that $C^* = (x^2 + y^2 - 1)^l (y - 1)^m = 0$ is an invariant algebraic curve with cofactor K. If $D = C - C^* \neq 0$, then $D = 0$ is also an invariant algebraic curve with the cofactor K. But D has degree $d \leqslant 2l + m - 2$. Again using lemma 9 we can assume that the highest order homogeneous term of D is of the form $D_d = (x^2 + y^2)^{l'} y^{m'}$ with $d = 2l' + m'$. Then, from the above proof we should have the linear part of K is $K_1 = (2l' + m'b)x + a(2l' + m')y$. It is in contradiction with the last paragraph. Hence, we must have $C = C^*$. This proves that for $b \neq 1 \pm \frac{1}{p}$ with $p \in \mathbb{N}$ and $a \in \mathbb{R} \backslash (\Upsilon \cup \{0\})$ system (7) has only the irreducible invariant algebraic curves $x^2 + y^2 = 1$ and $y = 1$. This proves the proposition. $\square$

Now we consider the exponential factors. We recall that if $F = \exp(G/H)$ is an exponential factor of system (1) with cofactor L, then $H = 0$ is an invariant algebraic curve of system (1) with a cofactor K_H, and G satisfies the following equation:

$$PG_x + QG_y = K_H G + LH. \tag{45}$$

For more details see [13].

Proposition 11. *For each $b \notin \mathbb{Q}$ there exists a numerable set $\Upsilon^* \supset \Upsilon$ such that if $a \in \mathbb{R} \backslash (\Upsilon^* \cup \{0\})$, then system (7) has no exponential factors.*

Proof. From proposition 10 system (7) has only the invariant algebraic curves $C_1 = x^2 + y^2 - 1 = 0$ and $C_2 = y - 1 = 0$. If system (7) has an exponential factor, we can assume that it has the form $F = \exp\left(\frac{G}{C_1^{l_1} C_2^{l_2}}\right)$ with a cofactor L, where l_1 and l_2 are non-negative integers. Since the invariant algebraic curve $C_1^{l_1} C_2^{l_2} = 0$ has the cofactor $K = l_1 K_1 + l_2 K_2 = 2l_1(x + ay) + l_2(bx + ay + a)$, from (45) we get that G satisfies the following equation:

$$[x^2 + axy + (1 - b)y^2 + by - 1]G_x + (bxy + ay^2 - bx - a)G_y$$
$$= [2l_1(x + ay) + l_2(bx + ay + a)]G + L(x^2 + y^2 - 1)^{l_1}(y - 1)^{l_2}. \tag{46}$$

Set $\deg G = n$. Since $\deg L \leqslant 1$, we can assume that $L = L_1 + L_0$, where L_i are homogeneous polynomials of degree i.

Case 1: $n + 1 < 2l_1 + l_2$. By equating the homogeneous terms of highest degree in (46) we obtain first that $L_1 = 0$, and after that $L_0 = 0$, and so $L = 0$. Thus G is an invariant algebraic curve. Moreover, from the assumption of this proposition we obtain that $G = cC_1^{l_1} C_2^{l_2}$, where c is a constant. Then, $F = $ constant, and it cannot be an exponential factor.

Case 2: $n + 1 = 2l_1 + l_2$. Then we have $L_1 = 0$. Set $G = \sum_{i=0}^{n} G_i(x, y)$ with G_i homogeneous polynomials of degree i and $G_n = \sum_{i=0}^{n} a_i x^{n-i} y^i$, where a_i are constants. Then, equating the

terms of (46) with degree $n + 1$ we get that

$$[x^2 + axy + (1 - b)y^2] \sum_{i=0}^{n}(n - i)a_i x^{n-i-1}y^i + (bxy + ay^2) \sum_{i=0}^{n} ia_i x^{n-i}y^{i-1}$$

$$= [2l_1(x + ay) + l_2(bx + ay)] \sum_{i=0}^{n} a_i x^{n-i}y^i + L_0(x^2 + y^2)^{l_1}y^{l_2}.$$

Using the relation $n + 1 = 2l_1 + l_2$ we can write this last equation as

$$\sum_{i=0}^{2l_1+l_2+1} \{[(b - 1)(i - l_2) - 1]a_i + aa_{i-1} + (1 - b)(2l_1 + l_2 + 1 - i)a_{i-2}\}x^{2l_1+l_2-i}y^i$$

$$= L_0 \sum_{i=0}^{l_1} \binom{l_1}{i} x^{2l_1-2i}y^{2i+l_2}$$

where $a_i = 0$ for $i < 0$ and $i > n$. The last equation is equivalent to

$$[(b - 1)(i - l_2) - 1]a_i + aa_{i-1} + (1 - b)(2l_1 + l_2 + 1 - i)a_{i-2} = 0$$
$$i = 0, 1, \ldots, l_2 - 1 \tag{47}$$

$$\sum_{j=0}^{2l_1+1} \left\{[(b - 1)j - 1]a_{j+l_2} + aa_{j+l_2-1} - (1 - b)(2l_1 + 1 - j)a_{j+l_2-2}\right\} x^{2l_1-j}y^{j+l_2}$$

$$= L_0 \sum_{i=0}^{l_1} \binom{l_1}{i} x^{2l_1-2i}y^{2i+l_2}. \tag{48}$$

Since $b \neq 1 \pm \frac{1}{k}$ for $k \in \mathbb{N}$, from (47) we get that $a_i = 0$ for $i = 0, 1, \ldots, l_2 - 1$. From (48) we obtain that for $i = 0, 1, \ldots, l_1$

$$[(b - 1)(2i + 1) - 1]a_{2i+1+l_2} + aa_{2i+l_2} - (1 - b)(2l_1 - 2i)a_{2i+l_2-1} = 0$$

$$[(b - 1)2i - 1]a_{2i+l_2} + aa_{2i+l_2-1} - (1 - b)(2l_1 + 1 - 2i)a_{2i+l_2-2} = L_0 \binom{l_1}{i}. \tag{49}$$

Solving (49) for $i = 0, 1, \ldots, l_1 - 1$ and its second equation with $i = l_1$ we get that

$$a_{l_2+h} = \bar{\mathcal{B}}_h(a)L_0 \qquad k = 0, 1, \ldots, 2l_1$$

where $\bar{\mathcal{B}}_h(a)$ is a polynomial of degree h in a whose coefficients are rational functions in b and l_1. The highest order term of $\bar{\mathcal{B}}_h(a)$ in a is $-1/\prod_{j=0}^{h}[(1 - b)j + 1]$. So the first equation of (49) with $i = l_1$ is $a\bar{\mathcal{B}}_{2l_1}L_0 = 0$. Since the coefficient of L_0 is a polynomial of degree $m + 1$, there exist at most $m + 1$ values of a such that it is equal to zero. We denote by $\bar{\gamma}_{l_1}$ the set of such a. Hence, if $a \notin \bar{\gamma}_{l_1}$ we must have $L_0 = 0$. This means that $L = 0$. So, system (7) has no exponential factors for $a \notin \bar{\gamma}_{l_1}$.

Case 3: $n = 2l_1 + l_2$. Let $L_1 = L_{10}x + L_{01}y$. Using the notations for G and G_n introduced in the study of case 2, equating the terms of (46) with degree $n + 1$ and doing some computations we get that

$$\sum_{i=0}^{n+2}[(b - 1)(i - l_2)]a_i + (1 - b)(n + 2 - i)a_{i-2}]x^{n+1-i}y^i$$

$$= L_{10} \sum_{i=0}^{l_1} \binom{l_1}{i} x^{2l_1-2i+1}y^{2i+l_2} + L_{01} \sum_{i=0}^{l_1} \binom{l_1}{i} x^{2l_1-2i}y^{2i+l_2+1}$$

where $a_i = 0$ for $i < 0$ and $i > n$. These equations are equivalent to

$$(i - l_2)a_i - (n + 2 - i)a_{i-2} = 0, \qquad i = 0, 1, \ldots, l_2 - 1$$

$$(b - 1)2ja_{2j+l_2} + (1 - b)(2l_1 + 2 - 2j)a_{2j+l_2-2} = L_{10}\binom{l_1}{j} \tag{50}$$

$$(b - 1)(2j + 1)a_{2j+l_2+1} + (1 - b)(2l_1 + 1 - 2j)a_{2j+l_2-1} = L_{01}\binom{l_1}{j}$$

where $j = 0, 1, \ldots, l_1$.

From the first equation of (50) we obtain that $a_i = 0$ for $i = 0, 1, \ldots, l_2 - 1$. Hence, the first equation of (50) with $j = 0$ induces to $L_{10} = 0$. Thus, we have

$$a_{2j+l_2} = \frac{2l_1 + 2 - 2j}{2j}a_{2j+l_2-2} \qquad j = 1, \ldots, l_1.$$

i.e. $a_{l_2+2j} = \binom{l_1}{j}a_{l_2}$, $j = 1, \cdots l_1$. From the second equation of (50) with $j = 0, 1, \ldots, l_1 - 1$ and using induction, we can prove that

$$a_{2j+1+l_2} = \frac{\mu_j}{b-1}L_{01} \qquad j = 0, 1, \ldots, l_1 - 1$$

with $\mu_j > 0$. Now the second equation of (50) with $j = l_1$ can be written as $(1 + \mu_{l_1})L_{01} = 0$. This implies that $L_{01} = 0$. Moreover, we have $a_{2j+1+l_2} = 0$ for $j = 0, 1, \ldots, l_1 - 1$.

From the above calculations we get that $L = L_0$ and

$$G_n = \sum_{i=0}^{n} a_i x^{n-i} y^i = a_{l_2}(x^2 + y^2)^{l_1} y^{l_2}.$$

Since $a_{l_2} \neq 0$, without loss of generality we assume that $a_{l_2} = 1$.

Equating the terms of (46) with degree n we get that

$$[x^2 + axy + (1 - b)y^2]G_{n-1,x} + (bxy + ay^2)G_{n-1,y} = [2l_1(x + ay) + l_2(bx + ay)]G_{n-1}$$
$$- byG_{nx} + bxG_{ny} + l_2aG_n + L_0(x^2 + y^2)^{l_1}y^{l_2}.$$

Let $G_{n-1} = \sum_{i=0}^{n-1} b_i x^{n-1-i} y^i$. Substituting G_{n-1} into the above equation and doing some computations, we can obtain that

$$\sum_{i=0}^{n}\{[l_2 - 1 - i + b(i - l_2)]b_i - ab_{i-1} + (1 - b)(n + 1 - i)b_{i-2}\}x^{n-i}y^i$$

$$= bl_2\sum_{i=0}^{l_1}\binom{l_1}{i}x^{2l_1+1-2i}y^{l_2+2i-1} + (l_2a + L_0)\sum_{i=0}^{l_1}\binom{l_1}{i}x^{2l_1-2i}y^{l_2+2i}$$

where $b_i = 0$ for $i < 0$ and $i > n - 1$. From this equation we obtain that

$$[l_2 - 1 - j + b(j - l_2)]b_j - ab_{j-1} + (1 - b)(n + 1 - j)b_{j-2} = 0$$
$$j = 0, 1, \ldots, l_2 - 2$$

$$[-2i + b(2i - 1)]b_{2i+l_2-1} - ab_{2i+l_2-2} + (1 - b)(2l_1 + 2 - 2i)b_{2i+l_2-3} = bl_2\binom{l_1}{i} \tag{51}$$

$$[-2i - 1 + 2bi]b_{2i+l_2} - ab_{2i+l_2-1} + (1 - b)(2l_1 + 1 - 2i)b_{2i+l_2-2} = (l_2a + L_0)\binom{l_1}{i}$$

with $i = 0, 1, \ldots, l_1$.

From the first equation of (51) we can prove that $b_j = 0$ for $j = 0, 1, \ldots, l_2 - 2$. From (51) with $i = 0, 1, \ldots, l_1 - 1$ and its first equation with $i = l_1$, working in a similar way to the proof of proposition 10 we can prove that

$$b_{l_2+2i-1} = \tilde{\mathcal{B}}_{2i-1}(a)L_0 - l_2 \binom{l_1}{i} \qquad b_{l_2+2i} = \tilde{\mathcal{B}}_{2i}(a)L_0 \qquad i = 0, 1, \ldots, l_1$$

where $\tilde{\mathcal{B}}_k(a)$ is a polynomial of degree k in a whose coefficients are rational functions in b and l_1. Using the last equation of (51) with $i = l_1$ we get

$$\left[a\tilde{\mathcal{B}}_{2l_1-1} + (b-1)\tilde{\mathcal{B}}_{2l_1-2} + 1 \right] L_0 = 0.$$

For every given b and l_1 there exist at most $2l_1$ values of a for which $a\tilde{\mathcal{B}}_{2l_1-1} + (b-1)\tilde{\mathcal{B}}_{2l_1-2} + 1$ is equal to zero. We denote by $\tilde{\gamma}_{l_1}$ the set of such a. Then if $a \notin \tilde{\gamma}_{l_1}$, we have $L_0 = 0$. So for every b satisfying the assumption of the proposition, if $a \notin \cup\tilde{\gamma}_{l_1}$ system (7) has no exponential factors.

Case 4: $n > 2l_1 + l_2$. Using the notations of case 2 for G and G_n, from (46) we get that

$$[x^2 + axy + (1-b)y^2]G_{nx} + (bxy + ay^2)G_{ny} = [2l_1(x+ay) + l_2(bx+ay)]G_n.$$

Working in a similar way to the previous case we can prove that the coefficients a_i in G_n satisfy the following equations:

$$[n - i - 2l_1 + b(i - l_2)]a_i + a(n - 2l_1 - l_2)a_{i-1} + (1-b)(n + 2 - i)a_{i-2} = 0$$

with $i = 0, 1, \ldots, n + 1$. Since $b \notin \mathbb{Q}$, from these equations we obtain that $a_i = 0$. So, $G_n = 0$. This implies that system (7) has no exponential factors.

Summing up these four cases the proof of the proposition follows. $\qquad \square$

In this last step we prove that for each $b \notin \mathbb{Q}$, if $a \in \mathbb{R} \backslash (\Upsilon^* \cup \{0\})$ system (7) is not Darboux integrable.

Suppose that the assumptions of proposition 11 are satisfied. Then, by propositions 10 and 11 we get that system (7) has only the invariant algebraic curves $x^2 + y^2 = 1$ with cofactor $K_1 = 2(x + ay)$ and $y = 1$ with cofactor $K_2 = bx + ay + a$, and has no exponential factors. It is easy to check that under these assumptions do not exist $\lambda_1, \lambda_2 \in \mathbb{C}$ not all zero such that $\lambda_1 K_1 + \lambda_2 K_2 = 0$ or $\lambda_1 K_1 + \lambda_2 K_2 = -\text{div}(P, Q) = -(2 + b)x - 3ay$, where $P = y(ax - by + b) + x^2 + y^2 - 1$ and $Q = bx(y - 1) + a(y^2 - 1)$. Hence, from the Darboux theory of integrability (see, for instance, [9] or [13]) it follows that system (7) is not Darboux integrable. We have finished the proof of theorem 3.

5. Proof of corollary 4.

Since $y = 1$ is invariant by system (8), after the change of variables

$$\bar{x} = \frac{x}{y-1} \qquad \bar{y} = \frac{y}{y-1} \qquad t = \frac{\tau}{y-1}$$

system (8) becomes the form of system (7), i.e.

$$\frac{d\bar{x}}{d\tau} = \bar{y}(a\bar{x} - b\bar{y} + b) + \bar{x}^2 + \bar{y}^2 - 1 = \bar{P}(x, y)$$

$$\frac{d\bar{y}}{d\tau} = b\bar{x}(\bar{y} - 1) + a(\bar{y}^2 - 1) = \bar{Q}(x, y). \tag{52}$$

Let $C(x, y)$ be a polynomial of degree n, and set $\bar{C}(\bar{x}, \bar{y}) = (\bar{y} - 1)^n C\left(\frac{\bar{x}}{\bar{y}-1}, \frac{\bar{y}}{\bar{y}-1}\right)$. We claim that if $C(x, y) = 0$ is an invariant algebraic curve of system (8) with cofactor $K(x, y)$

and $\bar{C} \not\equiv$ constant, then $\bar{C}(\bar{x}, \bar{y}) = 0$ is an invariant algebraic curve of system (52) with cofactor

$$\bar{K} = (\bar{y} - 1) K \left(\frac{\bar{x}}{\bar{y} - 1}, \frac{\bar{y}}{\bar{y} - 1} \right) + n \frac{\bar{Q}}{\bar{y} - 1}.$$

Indeed, straightforward calculations show that

$$\begin{aligned}
\bar{P}\bar{C}_{\bar{x}} + \bar{Q}\bar{C}_{\bar{y}} &= (\bar{y} - 1)^n \left[\frac{\bar{P}(\bar{y} - 1) - \bar{Q}\bar{x}}{(\bar{y} - 1)^2} C_x - \frac{\bar{Q}}{(\bar{y} - 1)^2} C_y + n \frac{\bar{Q}}{\bar{y} - 1} C \right] \\
&= (\bar{y} - 1)^n \left[(\bar{y} - 1) P C_x + (\bar{y} - 1) Q C_y + n \frac{\bar{Q}}{\bar{y} - 1} C \right] \\
&= (\bar{y} - 1)^{n+1} KC + n(\bar{y} - 1)^n \frac{\bar{Q}}{\bar{y} - 1} C = \bar{K}\bar{C}.
\end{aligned}$$

This proves the claim.

Now we claim that if $F(x, y) = \exp\left(\frac{G(x,y)}{H(x,y)} \right)$ is an exponential factor of system (8) with cofactor $L(x, y)$, then

$$\bar{F}(\bar{x}, \bar{y}) = \exp\left(G\left(\frac{\bar{x}}{\bar{y} - 1}, \frac{\bar{y}}{\bar{y} - 1} \right) \Big/ H\left(\frac{\bar{x}}{\bar{y} - 1}, \frac{\bar{y}}{\bar{y} - 1} \right) \right)$$

is an exponential factor of system (52) with cofactor $\bar{L}(\bar{x}, \bar{y}) = (\bar{y} - 1) L\left(\frac{\bar{x}}{\bar{y}-1}, \frac{\bar{y}}{\bar{y}-1} \right)$. In fact, we have

$$\begin{aligned}
\bar{P}\bar{F}_{\bar{x}} + \bar{Q}\bar{F}_{\bar{y}} &= \exp\left(\frac{G}{H} \right) H^{-2} \left[\left(\frac{\bar{P}(\bar{y} - 1) - \bar{Q}\bar{x}}{(\bar{y} - 1)^2} G_x - \frac{\bar{Q}}{(\bar{y} - 1)^2} G_y \right) H \right. \\
&\quad \left. - \left(\frac{\bar{P}(\bar{y} - 1) - \bar{Q}\bar{x}}{(\bar{y} - 1)^2} H_x - \frac{\bar{Q}}{(\bar{y} - 1)^2} H_y \right) G \right] \\
&= (\bar{y} - 1) \exp\left(\frac{G}{H} \right) H^{-2} [(PG_x + QG_y)H - (PH_x + QH_y)G] \\
&= (\bar{y} - 1)(PF_x + QF_y) \\
&= (\bar{y} - 1) LF = (\bar{y} - 1) L\left(\frac{\bar{x}}{\bar{y} - 1}, \frac{\bar{y}}{\bar{y} - 1} \right) \bar{F}.
\end{aligned}$$

This proves the claim.

From these two claims and the proof of theorem 3 we obtain that there exist values of a and b for which systems (8) and (7) have only two irreducible invariant algebraic curves and no exponential factors. Hence, for such values of $a \neq 0$ and b system (8) is not Darboux integrable. Otherwise, system (7) would have a Darboux integral, in contradiction with theorem 3. Hence, the proof of corollary 4 is complete.

Acknowledgments

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